Reconstruction of Gaussian Realizations by a Non-Optimal Algorithm Based on the Clipping

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Abstract. This document analyses a non-optimal algorithm for the Sampling-Reconstruction Procedure of Gaussian realizations. The algorithm is based on the clipping. It means that it knows just the zero crossings in the realization. To finding out its effectiveness, it is compared with an optimal algorithm, which considers some samples of the realization located at strategic points. The result is that the non-optimal algorithm does not give a correct reconstruction. So it is necessary to include a new parameter within this methodology to improve the performance. However, the application has some disadvantages, mainly reflected in the reconstruction error. For all this, it is possible to conclude that the non-optimal algorithms are just approximations to the optimal algorithms. The analysis is centered in the reconstruction of Markovian Gaussian realizations.

Keywords: Clipping, Conditional mean rule, Gaussian Markovian realization, Non optimal algorithm, Sampling-Reconstruction Procedure.

1 Introduction

A fundamental problem in communication theory is to establish a statistical description that defines the reconstruction of the realizations that compose a random process through the multitude of their samples. Throughout history there have been several investigations to try to resolve it. Perhaps the most important work was done by A. Balakrishnan [1]. But in recent years one methodology has been studied extensively by lots of people, it is called *conditional mean rule* (cmr). This rule is capable of reconstructing a random realization with the minimum error possible taking into account its main statistical characteristics. It gives to each random process its own *optimal reconstruction algorithm* and optimal reconstruction error algorithm (see for example [2-5]).

Despite having an optimal reconstruction algorithm, any realization of a random process has different ways to be reconstructed. That is, it can use an alternative methodology to the right methodology according to the characteristics of each process. This parallel technique can (or not) have the same amount of statistical parameters

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than the optimal technique uses to make the reconstruction. Although these parameters may not belong to the random process. Because of this, many times the reconstruction is not adequate. For this reason, it is called *non optimal reconstruction algorithm*. Even the conditional mean rule can be considered as non-optimal algorithm if it does not consider the appropriate parameters. The purpose is to have a simpler methodology in its development and implementation, considering the range of allowable error and the processing time.

In many cases, there is not enough information about the random process (or their realizations) to reconstruct it, specifically the value of the samples. Then, it is necessary to saturate the realization by a clipper converter. Getting a binary signal of the realization, which gives as information the moments where it crosses by zero.

The research is focus on the creation and analysis of the *Sampling-Reconstruction Procedure* (SRP) of Gaussian Markovian realizations having optimal and non-optimal algorithms. The methodologies are based on sampling methods using the zero crossings by clipping.

2 The Optimal Reconstruction Algorithm

The Sampling-Reconstruction Procedure using the optimal algorithm is performed by the conditional mean rule. The mean idea of this methodology has been proposed in [2]. Firstly, we consider a random process $x(t)$ characterized by its multidimensional probability functions $w_m[x(t_1), x(t_2), \dots, x(t_m)]$. One realization of this process is discretized in time instants $T = \{T_1, T_2, \dots, T_N\}$. Therefore, we form a set of samples $\mathrm{X}, T = x(T_1), x(T_2), \dots, x(T_N)$, where the number of samples N and their times of occurrence T are arbitrary. It means that the initial and central moment functions and their probability densities are modified. Now they are conditional and depend on the value of each sample $x(T_1), x(T_2), \dots, x(T_N)$.

In this way, the conditional mean function $\tilde{m}(t) = \langle x(t)|X, T\rangle$ is used as reconstruction function. The quality of the reconstruction is evaluated by the conditional variance function $\tilde{\sigma}(t) = \langle [x(t) - \tilde{m}(t)]^2|X, T\rangle$ or reconstruction error function. Both characteristics $\tilde{m}(t)$ and $\tilde{\sigma}(t)$ can be found on the basis of the conditional multidimensional probability density function $w_{N+1}(x(t)|X, T)$ of the given process. With these parameters, it is possible to reconstruct a random realization and get the quality of it. It is clear that one cannot know exactly the sampled realization, but with this rule we obtain a statistical approach for each moment of time t. The rule also provides the minimum estimation reconstruction error for random realizations with an arbitrary probability density function. For all that, the conditional mean rule is called optimal algorithm.

Considering that the realization to reconstruct is Gaussian, the conditional characteristics are [6-7]:

$$\tilde{m}(t) = m(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} K(t, T_i)\, a_{ij} \left[x(T_j) - m(T_j)\right] , \tag{1}$$

$$\tilde{\sigma}^2(t) = \sigma^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K(t, T_i)\, a_{ij}\, K(T_j, t)\ , \tag{2}$$

where $m(t)$ and $\sigma^2(t)$ are respectively the mathematical expectation and the variance of the initial process $x(t)$. $K(\cdot)$ is the covariance function, and a_{ij} represents the elements of the inverse covariance matrix. Assuming that the process is stationary, we consider $m(t) = 0$ and $\sigma^2(t) = 1$. This is the complete information about the process or realization given. According the properties of a Gaussian process, just the reconstruction function depends on the value of the samples; the reconstruction error function does not.

However, if we saturate the realization and apply the clipping operation, we have only the zero crossings as information. These zero crossings are considered samples. Under these conditions it is not possible to apply the conditional mean rule, because the result is zero. So we need to modify the methodology. This is accomplished by adding a sample between two zero crossings. With this proposal, we have the case where the sample is located at the midway between two zero crossings. Its magnitude is equal to the magnitude of the realization at this point.

Once we have the samples, it is possible to apply the condition mean rule for making the reconstruction and get a different result to zero. Then, the reconstruction function and the reconstruction error function are obtained by (1) and (2) respectively.

3 The Non Optimal Reconstruction Algorithm

The Sampling-Reconstruction Procedure using the non-optimal algorithm is also performed by the conditional mean rule. The difference consists in the samples that it uses. Although in this methodology is also used an additional sample between two zero crossing, the value of this sample is not considered part of the realization. The information we have is only the moments where the realization crosses by zero. Then, the sample used is located at the midpoint between the two zero crossings. But its magnitude is equal to the distance that exist between the two zero crossings where the samples is. It means that there is not relation with the realization. As in this case the conditional mean rule does not take into account the samples that belong to the realization, it is considered as a non-optimal algorithm.

When the zero crossings are obtained, and therefore the value of the additional samples, we can use the conditional mean rule to calculate the reconstruction function $\widehat{m}(t)$ of the non-optimal algorithm by (1). But the use of the reconstruction error function represented by (2) is not appropriate. Because of the non-optimal algorithm does not depend on the value of the samples, the equation (2) does not reflect the real reconstruction error. Besides, the error is the same that the obtained in the optimal algorithm because both algorithms have and depend on the same parameters. However, it is possible to define the quality of the reconstruction by a reconstruction total error approximate function $\varepsilon_T^2(t)$, which is directly related with the optimal reconstruction algorithm for knowing the differences that exist in the reconstructions of the realization. This function is realized in two parts. The first is a special deterministic part of the reconstruction error function generated by a relation between the reconstruction function of the optimal algorithm $\tilde{m}(t)$ and the reconstruction function of the non-optimal algorithm $\hat{m}(t)$, that is [8-9]:

$$\varepsilon_d^2(t) = [\tilde{m}(t) - \hat{m}(t)]^2 \ . \tag{3}$$

Clearly when $\tilde{m}(t) = \hat{m}(t)$ this error is equal to zero. The second is a random part of the reconstruction error. It is obtained on the basis of the reconstruction error function of the optimal algorithm $\tilde{\sigma}(t)$. Therefore, the reconstruction total error approximate function $\varepsilon_T^2(t)$ is determined by [8-9]:

$$\varepsilon_T^2(t) = \varepsilon_d^2(t) + \tilde{\sigma}^2(t) \ . \tag{4}$$

Now the error depends on the value of the samples as the reconstruction. So, its curves change drastically.

4 Comparison between both Reconstruction Algorithms

The random realization to reconstruct is obtained on the output of an one-stage RC filter when it is driven by Gaussian white noise. It means, it is a Gaussian Markovian realization. Its covariance function is:

$$K(\tau) = \exp(-\alpha|\tau|) \ . \tag{5}$$

Because we work with an unitary covariance time τ_c, then $\alpha = 1$. As example we consider a Gaussian Markovian realization composed by 61 samples in a time of 3 seconds, this is $N = 61$ and $t = 3$. The starting point is in zero, value of the mathematical expectation. The samples are separated periodically by 0.05 seconds, it means $\Delta T = 0.05$. The first step is to saturate the realization for knowing exactly where the zero crossings are located. In Fig. 1 are illustrated the realization to reconstruct and its saturated form.

For starting the reconstruction of the realization, both algorithms consider as samples the points where the zero crossings happen. They also include an additional sample located inside each pair of zero crossings. The optimal algorithm places the sample at the midpoint between the two zero crossings, with a magnitude that is equal to the magnitude of the realization at this point. That is, the samples are part of the realization. The non-optimal algorithm also places the sample at the midway between the two zero crossings. But the magnitude changes, now it is equal to the distance between the two zero crossings where the sample is located. It means, the samples do not belong to the realization. This is the difference between both algorithms; the optimal algorithm considers more statistical parameters which are part of the realization to reconstruct.

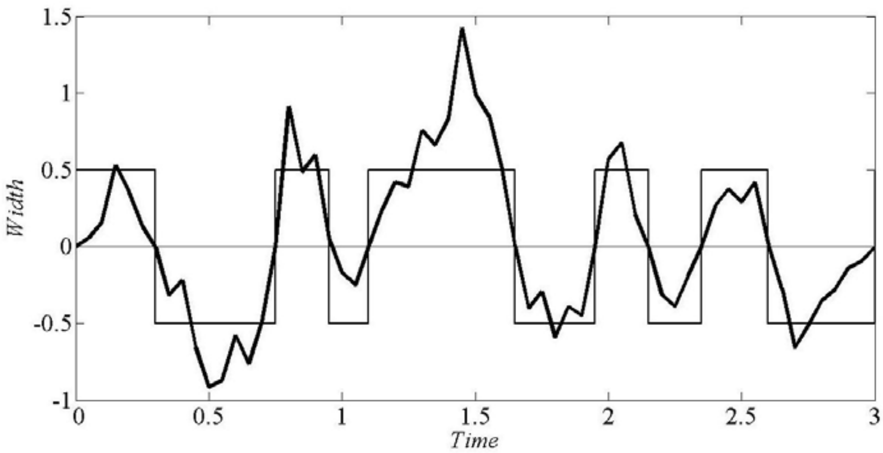

Fig. 1. Gaussian Markovian realization with $N = 61$ and $\Delta T = 0.05$, and its saturated form.

With the samples defined, we can perform the Sampling-Reconstruction Procedure of the realization. Both algorithms are based on the conditional mean rule for making the reconstruction function by (1). The difference takes place in the reconstruction error function. The optimal algorithm continues using that rule for determining the reconstruction error by (2). But the non-optimal algorithm associates the reconstruction functions of both algorithms for obtaining the reconstruction error, which is defined by (4). The optimal case does not consider the value of the samples to get the reconstruction error, the non-optimal case does. Although a feature of the Gaussian process is that the magnitude of the reconstruction error does not depend on their samples. In Fig. 2 are showed the reconstruction functions and the error reconstruction functions of both algorithms.

It is clear that the reconstruction curve originated by the optimal algorithm is better that the reconstruction curve caused by the non-optimal algorithm. This is because the first curve covers almost all the realization. The chaotic behavior of the realization and the low number of samples used for making the reconstruction ($N = 21$), prevent that the result could be better. Nevertheless, the curve in the optimal case is acceptable. Its error varies in each interval of time as a result of the different periods of separation between the samples. The maximum error is found in the middle of each interval, while the minimum error exists in the sampling points. The reconstruction curve of the non-optimal case does not cover a large part of the realization. One could even think that the behavior is another. This causes a bigger error. In some instants of time it is very high. It is important to note that the error is not equal to zero at the sampling points which are located between the zero crossings. When the difference between the reconstruction curves grows in those intervals, the error curve of the non-optimal case is further from a similar behavior to the error curve of the optimal case.

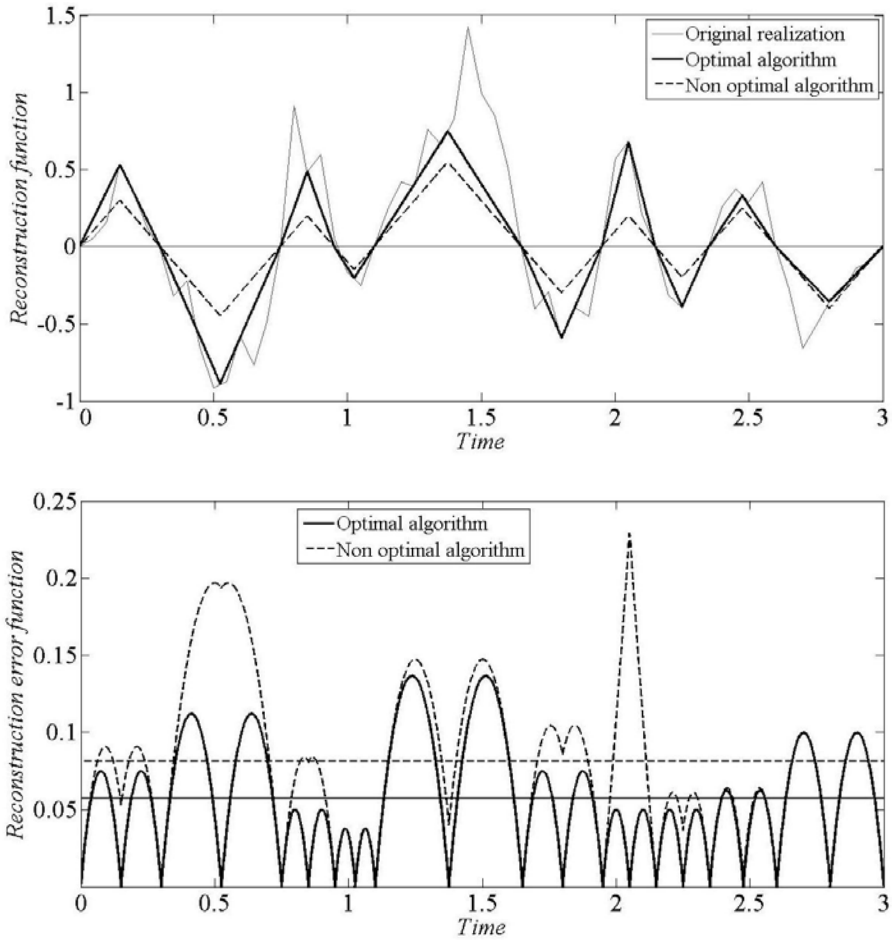

Fig. 2. Reconstruction and reconstruction error functions of a Gaussian Markovian realization.

Following the comparison, we include the average reconstruction error curves. They represent the area under the reconstruction error curves. They are obtained by:

$$\varepsilon_P^2(T) = \frac{1}{T}\int_0^T \rho^2(t)\,dt \ , \tag{6}$$

where $\rho^2(t)$ represents the reconstruction error according to each algorithm. Obviously, the average error in the non-optimal algorithm is always bigger than the average error in the optimal algorithm.

For improving the deficient results, it is possible to introduce a new parameter represented by k. The purpose of this constant is to multiply all samples used in the reconstruction function of the non-optimal algorithm for forming a new set of samples $X, T = k \times [x(T_1), x(T_2), \dots, x(T_N)]$. The value of k is given according to the width of the realization, and it can change until finding the best curve that represents the behavior of the realization. With k we could obtain a width in the reconstruction curve similar to the width of the realization. In Fig. 3 are presented the reconstruction functions and the reconstruction error functions of the non-optimal algorithm with different values of k, it is also presented the functions of the optimal algorithm.

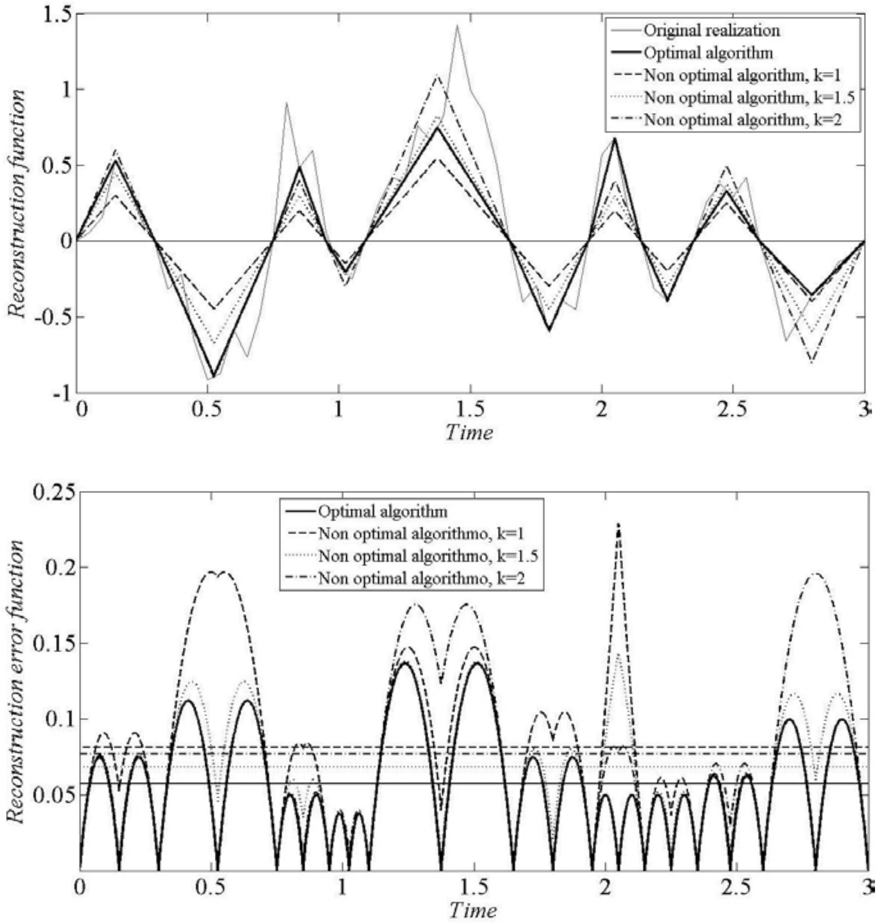

Fig. 3. Reconstruction and reconstruction error functions of a Gaussian Markovian realization with different values of k.

If we include the parameter k, the curves of the non optimal algorithm significantly improve. For example, when $k = 1.5$, the reconstruction curve maintains a close behavior to the realization. The new curve reaches to cover the realization much of the time. In many instants it is a curve very similar to the curve of the optimal algorithm. Hence, the reconstruction error is lower than when $k = 1$ in almost all the time. The same thing happens when we consider $k = 2$. Although with this value, the reconstruction error is higher in the time intervals where the separation between the samples is big. This is because the magnitude of the samples increases considerably. In both cases the results are better than when we do not use the parameter k. This is reflected in the average reconstruction error curves. In Fig. 4 there is a graph that indicates how the average reconstruction error is if we consider different values of k. It is also showed the relation with the optimal average reconstruction error.

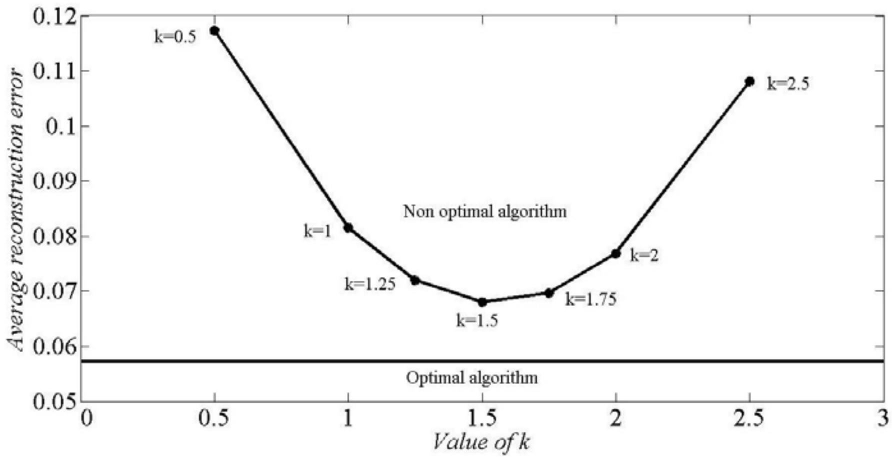

Fig. 4. Relation between the value of k and the magnitude of the average reconstruction error in the non-optimal algorithm.

One notes that for this example, the smallest reconstruction error is obtained when $k = 1.5$. But this may vary from one realization to another. Mainly because the characteristics change, such as: the number of zero crossings and the separation between them, the amplitude of the realization, the number of samples, and more. These cause that the graph of Fig. 4 also changes. Nevertheless after several experiments with many Gaussian realizations, we can conclude that using a value of k between 1.4 and 1.8, we get a similar reconstruction to the realization and therefore a smaller reconstruction error.

It is important to mention that the sampling procedure in the optimal algorithm is carried out at the transmitter. It means that on the output of the transmitter are sent three samples (two samples located at the zero crossings and one sample located at the midpoint between these zero crossings). But in the non-optimal algorithm the sampling procedure is performed at the receiver. Due to the transmitter gives only the location of the zero crossings as information, then the receiver is responsible for estimating the magnitude of the samples.

The figures demonstrate the difference between the reconstruction algorithms under analysis. The precise estimates of the reconstruction error function in the optimal algorithm give a family of curves that depends on several specific parameters of the realization, as the value of the samples. While the non-optimal algorithm gives curves that do not clearly describe the reality of the realization, because it considers samples that do not have relation with the samples of the realization. This could have the objective of making an analysis and procedure simpler and faster, in order to save time and space. But as a consequence, the reconstruction error is higher. In this way, we need to introduce a new parameter for improving the reconstruction. Obviously, the magnitude of the reconstruction error in the non-optimal algorithm is always higher in any case. This is because it is composed by an alternative methodology, and it is an approximation only. So it is a natural effect.

5 Conclusions

Two different reconstruction algorithms are analyzed to describe the Sampling-Reconstruction Procedure of Gaussian Markovian realizations. Both principal characteristics, reconstruction function and reconstruction error function, are obtained. The algorithms are based on techniques that use the sampling zero crossings by clipping. This is performed by saturating the realization through a clipper converter. Getting a binary signal of the realization, which gives as information the moments when the realization crosses by zero.

To make the reconstruction, the optimal algorithm adds an additional sample at the midpoint between each pair of zero crossings, with a magnitude equal to the magnitude of the realization at this point. The non-optimal algorithm also put an extra sample at the midpoint between two zero crossings, but now the magnitude is equal to the distance that exists between these zero crossings. The results show that the curves obtained for the optimal algorithm are widely better in all the time. This is reflected in a big difference between the reconstruction errors.

However, if we introduce a new parameter k that multiplies all samples of the reconstruction function in the non-optimal algorithm, we can get better curves that represent the behavior of the realization. Considering various realization, we conclude that the best results are obtained when k has a value between 1.4 and 1.8.

Clearly, the methodology that uses the greatest number of statistical parameters of the random realization, gives the correct reconstruction. This does not mean that the non-optimal algorithms are incorrect. Simply, they must be declared as special cases, which can be used depending on the application.

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